APJ ABDUL KALAM TECHNOLOGICAL UNIVERSITY

FIFTH SEMESTER B.TECH DEGREE EXAMINATION

EC 301 DIGITAL SIGNAL PROCESSING

Time: 3 Hours **Max. Marks : 100**

PART A

Question 1 is COMPULSORY and Answer EITHER Question 2 OR Question 3

Each Full Question Carries 15 Marks.

1.

a) Find the 4 point DFTs of two sequences g(n) and h(n) defined below, using a single 4 point DFT.
g(n)={1,2,0,1} and h(n)={2,2,1,1}. (8 Marks)

b) Prove if x(n) is a real valued sequence, then its DFT X(K) =X* (N-K).

(7 Marks)

2.

a) Find the DFT of a sequence x(n)={1,2,3,4,4,3,2,1}using radix-2 DIT algorithm. (8 Marks)

b) Find the IDFT of the sequence X(K)= {10,-2+2j,-2,-2-2j}.(7 Marks)

3.

a) In an LTI system the input sequence x(n)={1,1,1}and the impulse response h(n)={-1,1}.Find the response of the LTI system by using DFT –IDFT method. (8 Marks)

b) Derive the time reversal property of DFT. (7 Marks)

PARTB

Question 4 is COMPULSORY and Answer EITHER Question 5 OR Question 6

Each Full Question Carries 15 Marks

4.

a) Design an analog Butterworth filter that has a 2dB pass band attenuation at a frequency of 20 rad/sec and at least 10dB stop band attenuation at 30rad/sec (8 marks)

b) Compare FIR and IIR filters. (7 Marks)

5. a) Design a filter with $H_d(e^{jw}) = e^{-j3w}$, $-\pi/4 \le w \le \pi/4$

0 , $\pi/4 < |w| < \pi$

Using Hanning window with N=7 . (8 Marks)

b) Explain the relevance of window function and explain each window. (7 Marks)

6.

a) For the constraints $0.8 \le |H(e^{jw})| \le 1$, $0 \le w \le 0.2\pi$

$|H(e^{jw})| \le 0.2$, $0.6\pi \le w \le \pi$

With T=1sec.Determine system function H(z) for a Butterworth filter using impulse invariant method. (8 Marks)

b) Explain the bilinear transformation method of IIR filter design (7 Marks)

PART C

Question 7 is COMPULSORY and Answer EITHER Question 8 OR Question 9

Each Full Question Carries 20 Marks

7.

a) Obtain the direct form I, direct form II, cascade and parallel form realization for the system $y(n)= -0.1y(n-1)+0.2y(n-2)+3x(n)+3.6x(n-1)+0.6x(n-2)$. (12Marks)

b) filter is given by the system function $H(z)=1+(1/3)z^{-1}+(1/4)z^{-2}+(1/4)z^{-3}+(1/3)z^{-4}+z^{-5}$. Implement the filter with minimum number of multipliers. will the filter have linear phase characteristics? (8 Marks)

8. a) What are the factors involved with finite word length effects in digital filters. Explain any two effects in detail. (12 Marks)

b) Find the steady state variance of the noise in the output due to quantization of input for the first order filter $y(n)=ay(n-1)+x(n)$. (8 Marks)

9.

a) The output signal of an A/D converter is passed through a lowpass filter with transfer function is given by $H(z)=(1-a)z/(z-a)$ for $0<a<1$.Find the steady state output noise power due to quantization at the output of the digital filter. (12 Marks)

b) List out any two features of a fixed point processor that distinguishes it from a floating point processor.(8 Marks)

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FIFTH SEMESTER B.TECH DEGREE EXAMINATION.

COURSE CODE : EC 301

COURSE NAME : DIGITAL SIGNAL PROCESSING.

ANSWER KEY

PART A

1.

a. $x(n) = g(n) + jh(n) \quad 0 \le n \le 3.$

$x(n) = \{1+2j; 2+2j, 0+j, 1+j\}$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1+2j \\ 2+2j \\ 0+j \\ 1+j \end{bmatrix}$$

$$= \begin{bmatrix} 4+6j \\ 2 \\ -2 \\ 2j \end{bmatrix}$$

$$X(k) = \{4+6j,\ 2,\ -2,\ 2j\}$$

$$X^*(k) = \{4-6j,\ 2,\ -2,\ -2j\}$$

Hence $G(k) = \frac{1}{2}\left[X(k) + X^*(4-k)\right]$

$$= \frac{1}{2}\left[(4+j6,\ 2,\ -2,\ 2j) + (4-6j,\ -2j,\ -2,\ 2)\right]$$

$$= \{4,\ 1-j,\ -2,\ 1+j\}$$

and $H(k) = \frac{1}{2j}\left[X(k) - X^*(4-k)\right]$

$$= \frac{1}{2j}\left[(4+6j,\ 2,\ -2,\ 2j) - (4-6j,\ -2j,\ -2,\ 2)\right]$$

$$= \{6,\ 1-j,\ 0,\ 1+j\}$$

b. We know that

$$X(k) = DFT\{x(n)\}$$

$$= \sum_{n=0}^{N-1} x(n) W_N^{kn} \ ;\ k = 0, 1 \cdots N-1$$

Taking conjugates on both the sides, we get

$$X^*(k) = \sum_{n=0}^{N-1} x^*(n) W_N^{-kn}$$

Since $x(n)$ is real, we have $x^*(n) = x(n)$

$$\therefore X^*(k) = \sum_{n=0}^{N-1} x(n) W_N^{-kn}$$

$$\Rightarrow X^*(k) = \sum_{n=0}^{N-1} x(n) W_N^{-kn} \cdot W_N^{Nn} \quad (\text{since } W_N^{Nn} = 1)$$

$$= \sum_{n=0}^{N-1} x(n) W_N^{(N-k)n}$$

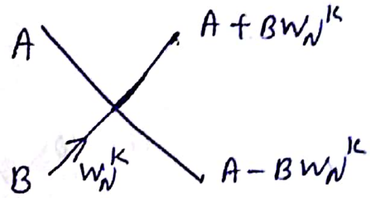
$$= X(N-k).$$

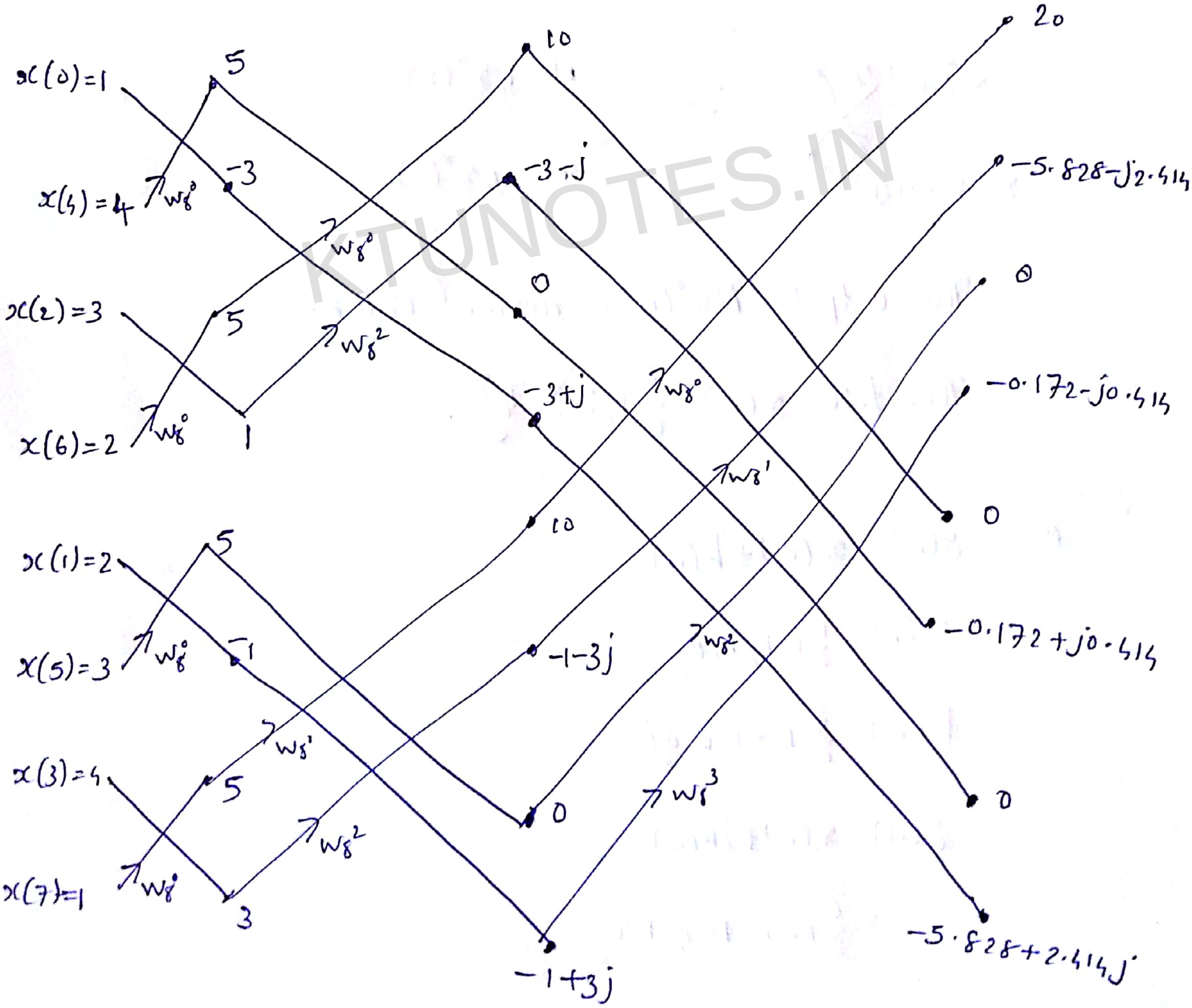
2. a. The twiddle factors associated with the flow graph are

$$W_8^0 = 1, \quad W_8^1 = \left(e^{-j2\pi/8}\right)^1 = e^{-j\pi/4} = 0.707 - j0.707$$

$$W_8^2 = \left(e^{-j2\pi/8}\right)^2 = e^{-j\pi/2} = -j$$

$$W_8^3 = \left(e^{-j2\pi/8}\right)^3 = e^{-j3\pi/4} = -0.707 - j0.707.$$

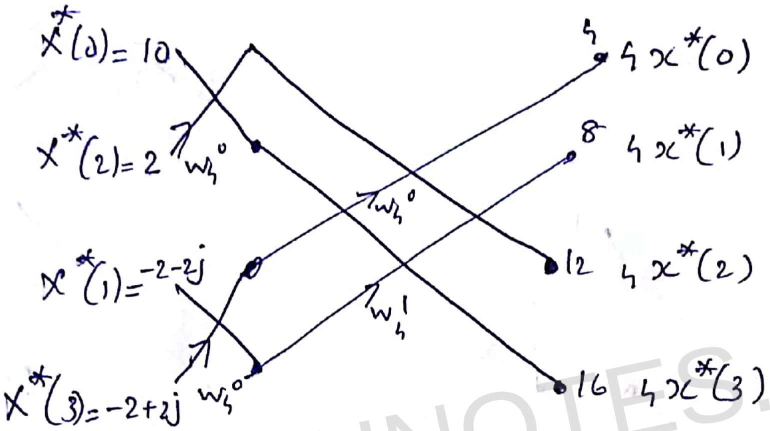




$$X(K) = \{20, -5.828 - j2.414, 0, -0.172 - j0.414, 0, -0.172 + j0.414, 0, -5.828 + j2.414\}$$

b. The twiddle factors are,

$$W_4^0 = 1, \quad W_4^1 = -j$$



The output $Nx^*(n)$ is normal order.

Therefore $x(n) = \{1, 2, 3, 4\}$.

3.

a. $y(n) = x(n) * h(n)$

$x(n) = \{1, 1, 1, 0\}$

$h(n) = \{-1, -1, 0, 0\}$

$g(n) = x(n) + jh(n)$

$= \{1-j, 1-j, 1, 0\}$

$$G(k) = \sum_{n=0}^{N-1} g(n) e^{-j2\pi/N nk}$$

$$= \sum_{n=0}^{3} g(n) e^{-j\pi/2 nk}$$

$$G(0) = \sum_{n=0}^{3} g(n) e^{0} = 1-j+1-j+1 = \underline{\underline{3-2j}}$$

$$G(1) = \sum_{n=0}^{3} g(n) e^{-j\pi/2 n} = \underline{\underline{-1-2j}}$$

$$G(2) = \sum_{n=0}^{3} g(n) e^{-j\pi n} = 1$$

$$G(3) = \sum_{n=0}^{3} g(n) e^{-j3\pi/2 n} = 1$$

$$G(k) = \{3-2j, -1-2j, 1, 1\}$$

$$G^*(k) = \{3+2j, -1+2j, 1, 1\}$$

$$G^*(N-k) = \{3+2j, 1, 1, -1+2j\}$$

$$X(k) = \frac{1}{2}\left[G(k) + G^*(N-k)\right]$$

$$= \{3, -j, 1, j\}$$

$$H(k) = \frac{1}{2j}\left[G(k) - G^*(N-k)\right]$$

$$= \{-2, -1+j, 0, -1-j\}$$

$$Y(k) = X(k) \cdot H(k)$$

$$= \{-6, 1+j, 0, 1-j\}$$

$$y(n) = IDFT[Y(k)]$$

$$= \frac{1}{4} \sum_{k=0}^{3} Y(k) e^{j\frac{\pi}{2}nk}$$

$$y(0) = \frac{1}{4}[-6+1+j-j+1] = -1$$

$$y(1) = \frac{1}{4} \sum_{k=0}^{3} Y(k) e^{j\frac{\pi}{2}k} = -2$$

$$y(2) = \frac{1}{4} \sum_{k=0}^{3} Y(k) e^{j\pi} = -2$$

$$y(3) = \frac{1}{4} \sum_{k=0}^{3} Y(k) e^{j3\frac{\pi}{2}k} = -1$$

$$y(n) = \{-1, -2, -2, -1\}$$

b.

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk}$$

Substituting $m = N-n \Rightarrow n = N-m$

$$X(k) = \sum_{m=N}^{1} x(N-m) W_N^{(N-m)k}$$

Replacing m by n

$$X(k) = \sum_{m=N}^{1} x(N-n) W_N^{(N-n)k}$$

$$X(k) = \sum_{n=0}^{N-1} x(N-n) W_N^{Nk} \cdot W_N^{-nk} \qquad | W_N^{Nk} = 1$$

$$X(N-k) = \sum_{n=0}^{N-1} x(N-n) W_N^{-n(N-k)}$$

$$= \sum_{n=0}^{N-1} x(N-n) W_N^{-nN} \cdot W_N^{nk} \qquad | W_N^{-nN} = 1$$

$$\therefore X(N-k) = \sum_{n=0}^{N-1} x(N-n) W_N^{nk}.$$

4.

a. Given $\alpha_p = 2dB$, $\Omega_p = 20 rad/sec$.

$\alpha_s = 10 dB$, $\Omega_s = 30 rad/sec$

$$N \geq \frac{\log\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log\frac{\Omega_s}{\Omega_p}}$$

$$\geq \frac{\log\sqrt{\frac{10-1}{10^{0.2}-1}}}{\log\frac{30}{20}}$$

$$\geq 3.37.$$

Rounding off N to the next highest integer we get,

$N = 4$

The normalized lowpass Butterworth filter for $N=4$ can be written as,

$$H(s) = \frac{1}{(s^2+0.76537s+1)(s^2+1.8477s+1)}$$

$$\Omega_c = \frac{\Omega_p}{(10^{0.1\alpha_p}-1)^{1/2N}} = \frac{20}{(10^{0.2}-1)^{1/8}} = \underline{\underline{21.3868}}$$

The transfer function for $\Omega_c = 21.3868$ can be Obtained by substituting,

$$s \rightarrow \frac{s}{21.3868} \text{ in } H(s)$$

$$\therefore H(s) = \frac{0.20921\times10^6}{(s^2+16.3686s+457.394)(s^2+39.5176s+457.394)}$$

b.

FIR filter	IIR filter
1) The impulse response of this filter is restricted to finite number of samples.	1. The impulse response of this filter extends over an infinite duration.
2. FIR filters can have precisely linear phase.	2. These filters do not have linear phase.
3. Closed-form design equations do not exist.	3. A variety of frequency selective filters can be designed using closed form design formulas.
4. Always stable.	4. Not always stable.
5. Errors due to roundoff noise are less severe.	5. IIR filters are more susceptible to errors due to round off noise.
6. Greater flexibility to control the shape of their magnitude response.	6. Less flexibility specially for obtaining non-standard frequency response.

5. a.

$$H_d(e^{j\omega}) = e^{-j3\omega}$$

The frequency response is having a term $e^{-j\omega(N-1)/2}$ which gives $h(n)$ symmetrical about $n = \frac{N-1}{2} = 3$, we get a causal sequence.

We have, $$h_d(n) = \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} e^{-j3\omega} \cdot e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} e^{j(n-3)\omega} d\omega$$

$$= \frac{\sin \frac{\pi}{4}(n-3)}{\pi(n-3)}$$

For $n=7$, we have

$h_d(0) = h_d(6) = 0.075$.

$h_d(1) = h_d(5) = 0.159$.

$h_d(2) = h_d(4) = 0.22$

$h_d(3) = 0.25$.

The non-causal window sequence is,

$$W_{Hn}(n) = 0.5 + 0.5\cos\frac{2\pi n}{N-1} \quad \text{for } -(N-1)/2 \le n \le \frac{N-1}{2}$$

$$= 0, \quad \text{otherwise.}$$

For $N=7$.

$$W_{Hn}(n) = 0.5 + 0.5\cos\frac{2\pi n}{N-1} \quad \text{for } -3 \le n \le 3$$

$$0, \quad \text{otherwise.}$$

$w_{Hn}(0) = 0.5 + 0.5 = 1$

$w_{Hn}(-1) = w_{Hn}(1) = 0.5 + 0.5\cos\frac{\pi}{3} = 0.75$

$w_{Hn}(-2) = w_{Hn}(2) = 0.5 + 0.5\cos\frac{2\pi}{3} = 0.25$

$w_{Hn}(-3) = 0.5 + 0.5\cos\pi = 0.$

The causal window sequence can be obtained by shifting the sequence $w_{Hn}(n)$ to right by 3 samples.

i.e. $w_{Hn}(0) = w_{Hn}(6) = 0$, $w_{Hn}(1) = w_{Hn}(5) = 0.25$

$w_{Hn}(2) = w_{Hn}(4) = 0.75$, $w_{Hn}(3) = 1$

The filter coefficients using Hanning window are,

$h(n) = h_d(n) w_{Hn}(0)$ for $0 \leq n \leq 6$.

$h(0) = h(6) = h_d(0) w_{Hn}(0) = (0.075)(0) = 0.$

$h(1) = h(5) = h_d(1) \cdot w_{Hn}(1) = (0.159)(0.25) = 0.03975$

$h(2) = h(4) = h_d(2) \cdot w_{Hn}(2) = (0.22)(0.75) = 0.165$

$h(3) = h_d(3) \cdot w_{Hn}(3) = (0.25)(1) = 0.25.$

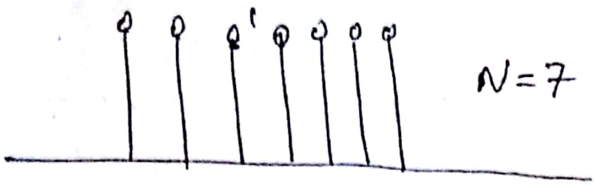
b) The window chosen for truncating the infinite impulse response have some desirable characteristics. They are

1) The central lobe of the frequency response of the window should contain most of the energy and should be narrow.
2. The highest side lobe level of the ~~h~~ frequency response should be small.

Rectangular window

rectangular window sequence is given by,

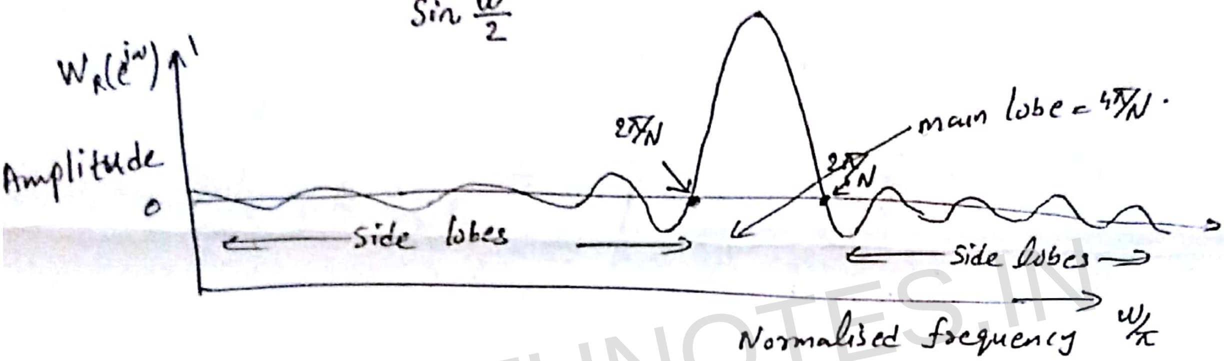
$$w_R(n) = 1 \text{ for } -\frac{(N-1)}{2} \leq n \leq \frac{(N-1)}{2}.$$
$$\quad\quad 0 \text{ otherwise.}$$



Rectangular window.

The spectrum of the rectangular window is given by,

$$W_R(e^{jw}) = \frac{\sin \frac{wN}{2}}{\sin \frac{w}{2}}.$$



The frequency response differs from the desired response in many ways. It does not follow quick transitions in the desired response. The desired response of a lowpass filter changes abruptly from passband to stopband, but the frequency response changes slowly. This region of gradual change is called filter's transition region, which is due to the convolution of the desired response with the window response's main lobe.

The convolution of the desired response and the window response's sidelobes gives rise to the ripples in the passband and stopband. The amplitude of the ripples is dictated by the amplitude of the sidelobes.

For the rectangular window, the amplitude of the sidelobes is unaffected by the length of the window. so increase in length N will not reduce the ripples, but increase its frequency. This effect where maximum ripple occurs just before and after the transition band is known as Gibbs phenomenon.

The Gibbs phenomenon can be reduced by using a less abrupt truncation of filter coefficients. This can be achieved using a window function that tapers smoothly towards zero at both ends. One such type of window is triangular window.

2) <u>The Triangular or Bartlett window.</u>

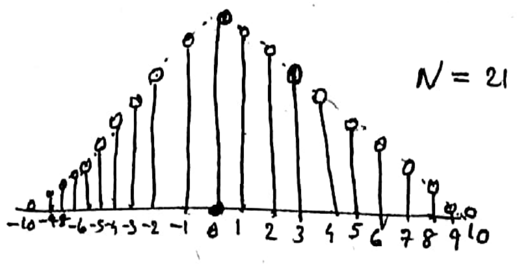
The N-point triangular window is given by,

$$w_T(n) = 1 - \frac{2|n|}{N-1} \quad \text{for } -\frac{(N-1)}{2} \le n \le \frac{(N-1)}{2}.$$

The fourier transform of the triangular window is,

$$W_T(e^{j\omega}) = \left(\frac{\sin\left(\frac{N-1}{4}\right)\omega}{\sin \frac{\omega}{2}} \right)^2$$

<u>Triangular window sequence.</u>



The triangular window produces a smooth magnitude responses in both passband and stopband. But it has the following disadvantages, when compared to magnitude response obtained by using rectangular window.

1. The transition region is more
2. The attenuation in stopband is less.

Raised Cosine Window

The raised cosine window multiplies the central fourier coefficients by approximately unity and smoothly truncate the fourier coefficients toward the ends of the filter. The window sequence is of the form,

$$w_\alpha(n) = \alpha + (1-\alpha)\cos\left(\frac{2\pi n}{N-1}\right), \text{ for } -\frac{(N-1)}{2} \le n \le \frac{(N-1)}{2}.$$

$$0 \quad , \text{otherwise}.$$

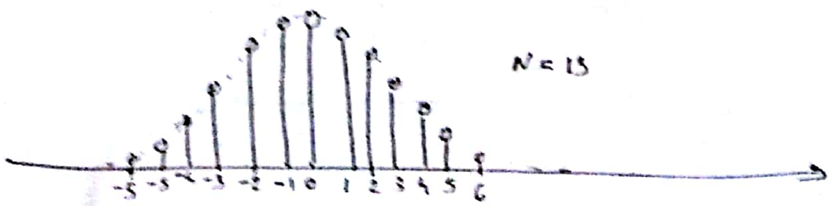
4. Hanning Window

The Hanning window sequence can be obtained by substituting $\alpha = 0.5$ in raised cosine window equation.

$$w_{Hn}(n) = 0.5 + 0.5\cos\left(\frac{2\pi n}{N-1}\right), \text{ for } -\frac{(N-1)}{2} \le n \le \frac{(N-1)}{2}.$$

$$0 \quad , \text{otherwise}$$

The main lobe width of Hanning window is twice that of the rectangular window, which results in a doubling of the transition region of the filter. The first side lobe of Hanning window spectrum is approximately one tenth that of the rectangular window. This results in smaller ripples in both passband and stopband of the lowpass filter designed using Hanning window. The minimum stopband attenuation of the filter is 44 dB which is 23 dB lower than the filter designed using rectangular window. At higher frequencies the stopband attenuation is even greater.



Hanning window sequence.

5. Hamming window.

The equation for Hamming window can be obtained by Substituting $\alpha = 0.54$ in raised Cosine window equation.

$$w_{Hm}(n) = 0.54 + 0.46 \cos\left(\frac{2\pi n}{N-1}\right), \text{ for } -(N-1)/2 \le n \le (N-1)/2.$$

$$0 \qquad , \text{ otherwise.}$$

The peak side lobe level is down about 41dB from the mainlobe peak, an improvement of 10dB relative to the Hanning window. The first side lobe peak is −53dB, an improvement of 9dB with respect to Hanning window filter. However, at higher frequencies the stopband attenuation is low when compared to that of Hanning window.

Because the Hamming window generates less oscillation in the side lobes than the Hanning window, for the same main lobe width, the Hamming window is generally preferred.

6) Blackman window.

$$w_B(n) = 0.42 + 0.5 \cos\left(\frac{2\pi n}{N-1}\right) + 0.08 \cos\left(\frac{4\pi n}{N-1}\right)$$

$$\text{for } -(N-1)/2 \le n \le (N-1)/2.$$

$$0, \quad \text{otherwise.}$$

The peak side lobe level is down about 57dB from the main lobe peak, an improvement of 16dB relative to the Hamming window. The side lobe attenuation of a lowpass filter using Blackman window is −74dB.

6.

a. Given $\frac{1}{\sqrt{1+\varepsilon^2}} = 0.8$ from which $\varepsilon = 0.75$

$\frac{1}{\sqrt{1+\lambda^2}} = 0.2$ from which $\lambda = 4.899.$

$\omega_s = 0.6\pi$ rad , $\omega_p = 0.2\pi$ rad

$$\frac{\omega_s}{\omega_p} = \frac{\Omega_s T}{\Omega_p T} = \frac{\Omega_s}{\Omega_p} = \frac{0.6\pi}{0.2\pi} = 3$$

$$N = \frac{\log \lambda/\varepsilon}{\log 1/k} = \frac{\log 4.899}{\log 3}$$

$$= 1.71$$

$\therefore N = \underline{\underline{2}}$

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

$$\Omega_c = \frac{\Omega_p}{\varepsilon^{1/N}} = \frac{0.2\pi}{(0.75)^{1/2}}$$

$$= \underline{0.231\pi}$$

$$H_a(s) = H(s)\Big|_{s \to s/0.231\pi}$$

$$= \frac{0.5266}{s^2 + 1.03s + 0.5266}.$$

$$= \frac{0.516j}{s + 0.51 + j0.51} - \frac{0.516j}{s + 0.51 - j0.51}$$

$$= \frac{0.516j}{s - (-0.51 - j0.51)} - \frac{0.516j}{s - (-0.51 + j0.51)}$$

$$H(z) = \frac{0.516j}{1 - e^{-0.51T} e^{-j0.51T} z^{-1}} - \frac{0.516j}{1 - e^{-0.51T} e^{j0.51T} z^{-1}}$$

$T = 1\,\text{sec}$

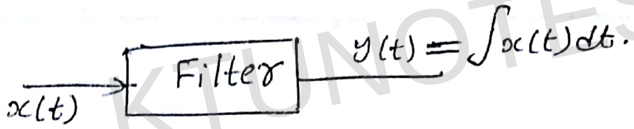
$$H(z) = \frac{0.3019 z^{-1}}{1 - 1.0482 z^{-1} + 0.36 z^{-2}}.$$

9b. Bilinear or Tustin Transformation.

To overcome the difficulty of impulse invariant method, bilinear transformation may be used. In this method the entire $j\Omega$ axis for $-\infty < \Omega < \infty$ maps uniquely onto the unit circle for $-\pi < \omega < \pi$.

The bilinear transform provides a nonlinear one to one mapping of the frequency points on the $j\Omega$ axis in the 's' plane to those on the unit circle in the 'z' plane, thus this procedure allows us to implement any kind of digital filters. The digital filter derived from this method has approximately the same time domain response as the original analog filter for any type of inputs.

Let us consider simple analog filter (analog integrator) as shown in figure,



Take L·T on both sides of y(t) we get,

$$Y(s) = \frac{1}{S}\left[X(s) - X(0)\right] \quad \text{. Assume } X(0)=0.$$

$$\text{Thus, } Y(s) = \frac{X(s)}{S}.$$

$$\text{or } H(s) = \frac{Y(s)}{X(s)} = 1/s. \quad \text{—— (1)}$$

Take L^{-1} on both sides of equ (1), we get,

$$h(t) = u(t) \quad ; \quad \text{for } |t| \geq 0.$$
$$= 0 \quad , \quad \text{elsewhere.} \quad \text{—— (2)}$$

The o/p response to an any arbitary input is determined by using convolution integral,

$$\text{ie; } y(t) - \int_0^t x(\tau)\, h(t-\tau)\, d\tau \;.. \quad \text{—— (3)}$$

The input x(t) is any arbitary input existing between

$t_1 \le t \le t_2$ ...

as,

$$y(t_2) - y(t_1) = \int_{t_1}^{t_2} x(\tau) u(t-\tau) d\tau. \quad — ④$$

We know, $u(t) = 1$, and $u(t-\tau) = 1$.

$$y(t_2) - y(t_1) = \int_{t_1}^{t_2} x(\tau) d\tau = \left(\frac{t_2 - t_1}{2}\right)\left[x(t_1) + x(t_2)\right].$$

or $y(t_2) - y(t_1) = \frac{T}{2}\left[x(t_1) + x(t_2)\right]$ where $T = t_2 - t_1$ ⑤

Apply the sampling procedure by substitute $t_1 = (n-1)T$, and $t_2 = nT = n$ then,

$$y(nT) - y[(n-1)T] = \frac{T}{2}\left[x(n-1)T + x(nT)\right].$$

or simply $y(n) - y(n-1) = \frac{T}{2}\left[x(n-1) + x(n)\right]$. — ⑥

Take Z·T on both sides of the above equation.

$$Y(z) - z^{-1} Y(z) = \frac{T}{2}\left[z^{-1} X(z) + X(z)\right].$$

$$Y(z)\left[1 - z^{-1}\right] = \frac{T}{2}\left[1 + z^{-1}\right] X(z).$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{T}{2}\left(\frac{1+z^{-1}}{1-z^{-1}}\right) \quad — ⑦$$

Equating equ ① & ⑦, $H(z) = H(s)$ thuse we get,

$$\frac{1}{s} = \frac{T}{2}\left(\frac{1+z^{-1}}{1-z^{-1}}\right) \text{ or}$$

$$\boxed{s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) = \frac{2}{T}\left(\frac{z-1}{z+1}\right)} \quad — ⑧$$

$$\boxed{\text{Thus } H(z) = H(s)\Big|_{s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}}$$

We know, $s = \frac{2}{T}\left(\frac{z-1}{z+1}\right)$

$$\frac{sT}{2} = \frac{z-1}{z+1}$$

$$sTz + sT = 2z - 2$$

$$sTz - 2z = -2 - sT$$

$$z(sT-2) = -(2+sT)$$

$$z = \frac{-(2+sT)}{sT-2}$$

$$= \frac{2+sT}{2-sT} = \frac{1+\frac{T}{2}s}{1-\frac{T}{2}s}.$$

If $s = \sigma + j\Omega$. then,

$$z = \frac{1+\frac{T}{2}(\sigma+j\Omega)}{1-\frac{T}{2}(\sigma+j\Omega)} = \frac{\left(1+\frac{T}{2}\sigma\right)+j\frac{T\Omega}{2}}{\left(1-\frac{T}{2}\sigma\right)-j\frac{T\Omega}{2}}$$

$$|z| = \left|\frac{\left(1+\frac{T}{2}\sigma\right)^2+\left(\frac{T}{2}\Omega\right)^2}{\left(1-\frac{T}{2}\sigma\right)^2+\left(\frac{T}{2}\Omega\right)^2}\right|^{1/2}$$

- When $Re(s) = \sigma > 0$ then $|z| > 1$
  $\sigma < 0$, then $|z| < 1$.
  $\sigma = 0$, then $|z| = 1$

Therefore the '$j\Omega$' axis maps on to the unit circle $|z|=1$, the LHS of s plane maps onto the interior part of the unit circle $|z|=1$ and the RHS of 's' plane maps onto the outside the unit circle $|z|=1$. As a result, of this "the stable analog filter is transformed into a stable digital filter".

and $z = e^{j\omega}$ in equ ⑧.

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$j\Omega = \frac{2}{T}\left(\frac{1-e^{-j\omega}}{1+e^{-j\omega}}\right) = \frac{2\cancel{e^{-j\omega/2}}}{T\cancel{e^{-j\omega/2}}}\left(\frac{e^{j\omega/2} - e^{-j\omega/2}}{e^{j\omega/2}+e^{-j\omega/2}}\right)$$

$$j\Omega = j\frac{2}{T}\tan\left(\omega/2\right)$$

$$\Omega = \frac{2}{T}\tan\left(\omega/2\right) \rightarrow (9)$$

$$\frac{\Omega T}{2} = \tan\left(\omega/2\right)$$

$$\tan^{-1}\left(\frac{\Omega T}{2}\right) = \frac{\omega}{2}$$

$$\omega = 2\tan^{-1}\left(\frac{\Omega T}{2}\right) \longrightarrow (10)$$

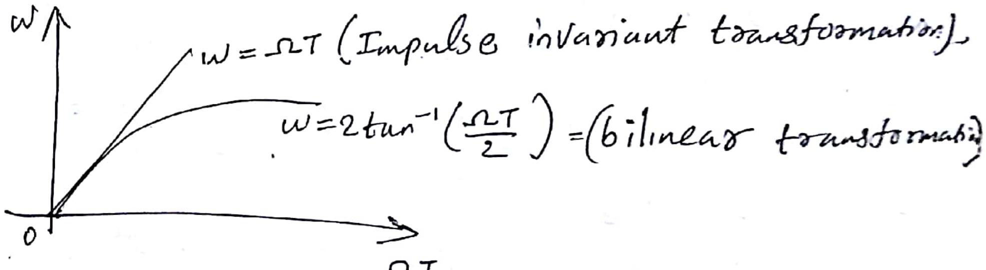
$$\frac{e^{x}-e^{-x}}{2j} = \sin x$$

$$\frac{e^{x}+e^{-x}}{2} = \cos x$$

$$\frac{1}{j}\left(\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}\right) = \tan x$$

$$\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}} = j\tan x$$

eq ⑨ and equ ⑩ shows that there is a nonlinear relationship between analog frequency (Ω) and digital frequency (ω).

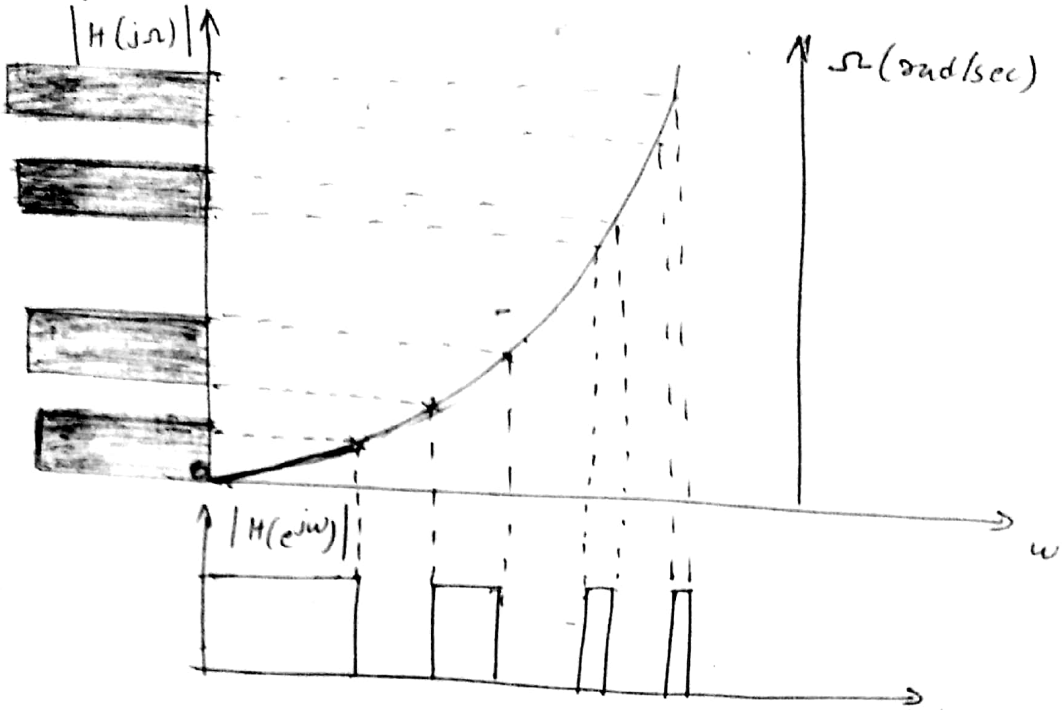


(Relation b/w 'Ω' and 'ω')

* **Warping effect.**

At low frequencies, the digital filter has the same frequency response as the reference analog filter. For higher frequencies, the relationship between 'Ω' and 'ω' is highly nonlinear and a distorti

...roduced in the frequency domain of the digital filter. (15)
... distortion is known as "frequency warping effect". The warping effect influence both the amplitude and phase response of the digital filter.



Influence of Warping effect on amplitude response.

→ Consider the analog filter with number of Pass band (Shaded region) centred at fixed intervals. The digital filter has the same number of passbands, but its centre frequencies and bandwidths of passbands is reduced disproportionately.

The warping effect can be eliminated by prewarping the analog filter by computing analog frequency.

Advantages of Bilinear transformation.

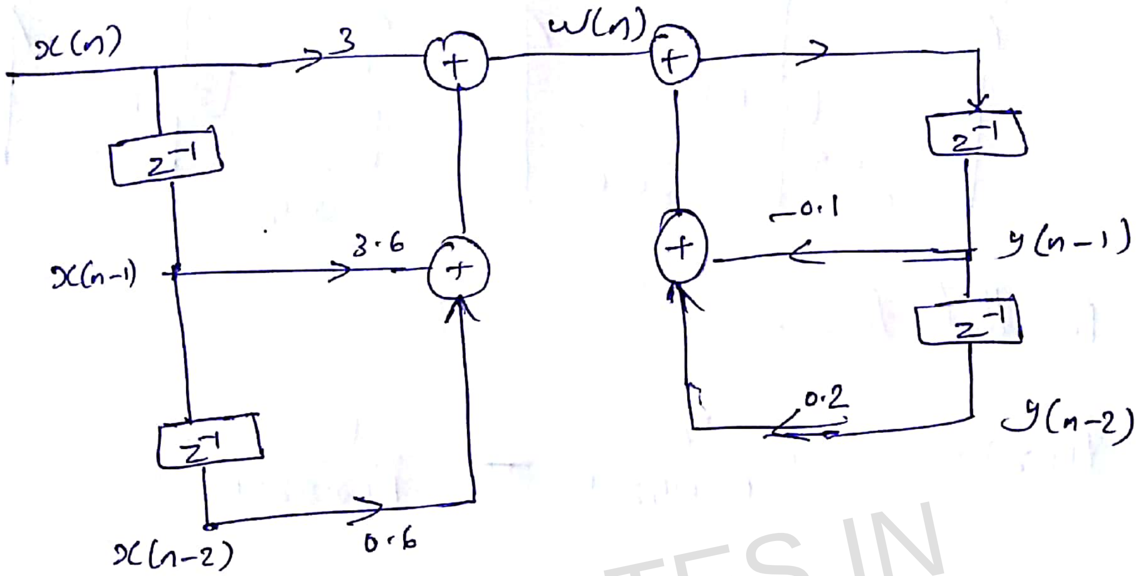
1. There is no aliasing, thus this transformation may be used to any filters.
2. It provides one to one mapping.
3. Stable analog filters can be mapped into realizable Stable digital filter.
4. The effect of warping on amplitude response can be eliminated by prewarping the analog filter.

7 a.

Direct form I

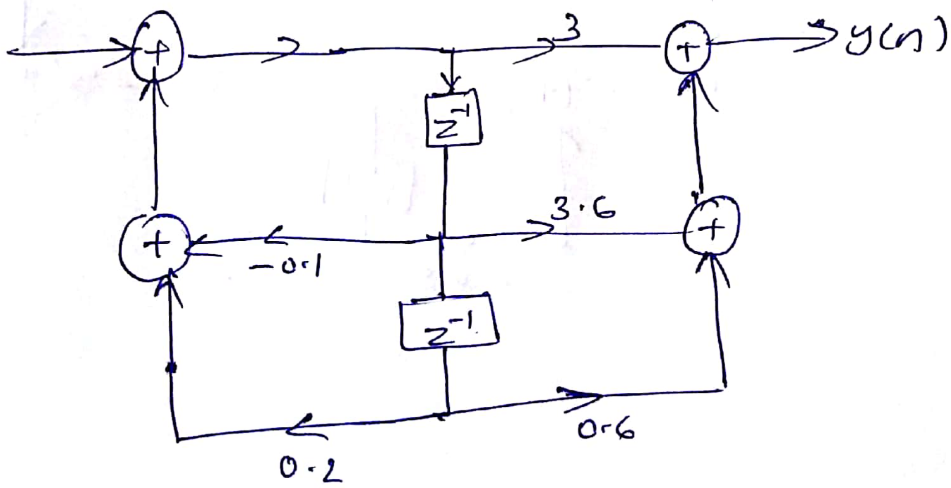
Let $3x(n) + 3.6\,x(n-1) + 0.6\,x(n-2) = w(n)$

$y(n) = -0.1y(n-1) + 0.2y(n-2) + w(n)$.



Direct form II

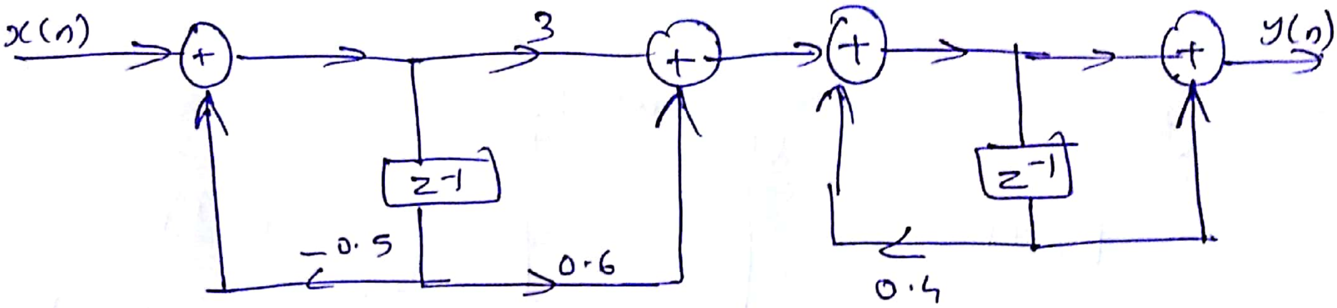
$$H(z) = \frac{Y(z)}{X(z)} = \frac{3 + 3.6z^{-1} + 0.6z^{-2}}{1 + 0.1z^{-1} - 0.2z^{-2}}$$



Cascade form-

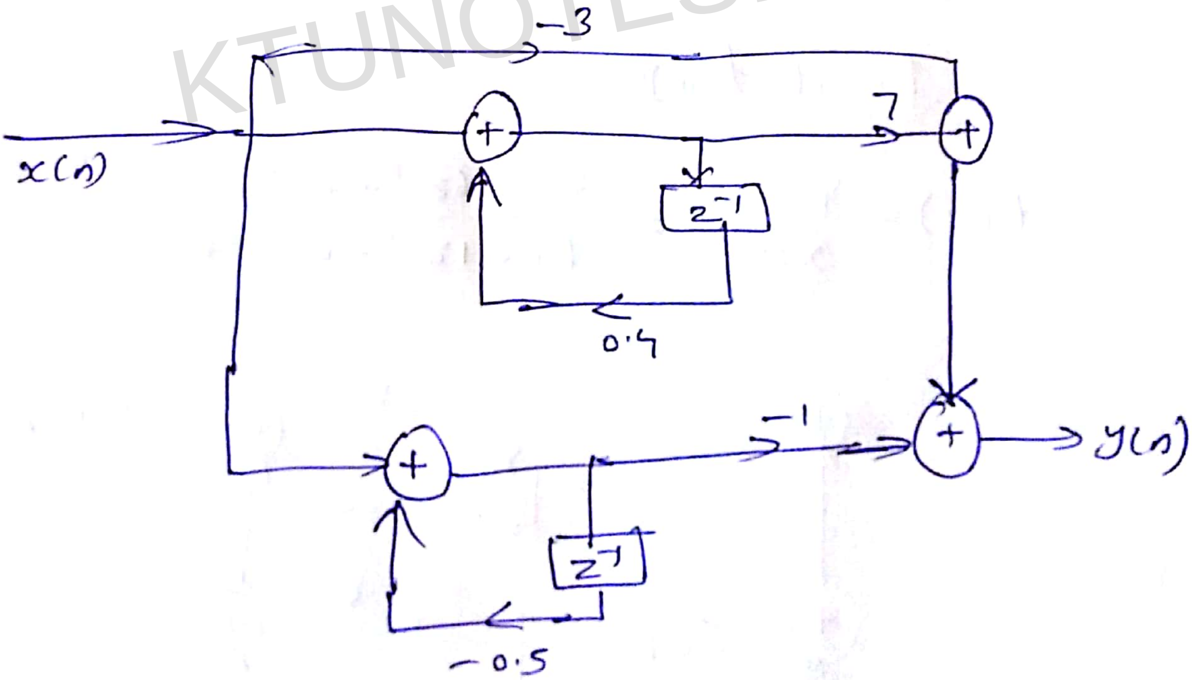
$$H(z) = \frac{(3+0.6z^{-1})(1+z^{-1})}{(1+0.5z^{-1})(1-0.4z^{-1})}$$

$$H_1(z) = \frac{3+0.6z^{-1}}{1+0.5z^{-1}} \quad \text{and} \quad H_2(z) = \frac{1+z^{-1}}{1-0.4z^{-1}}.$$



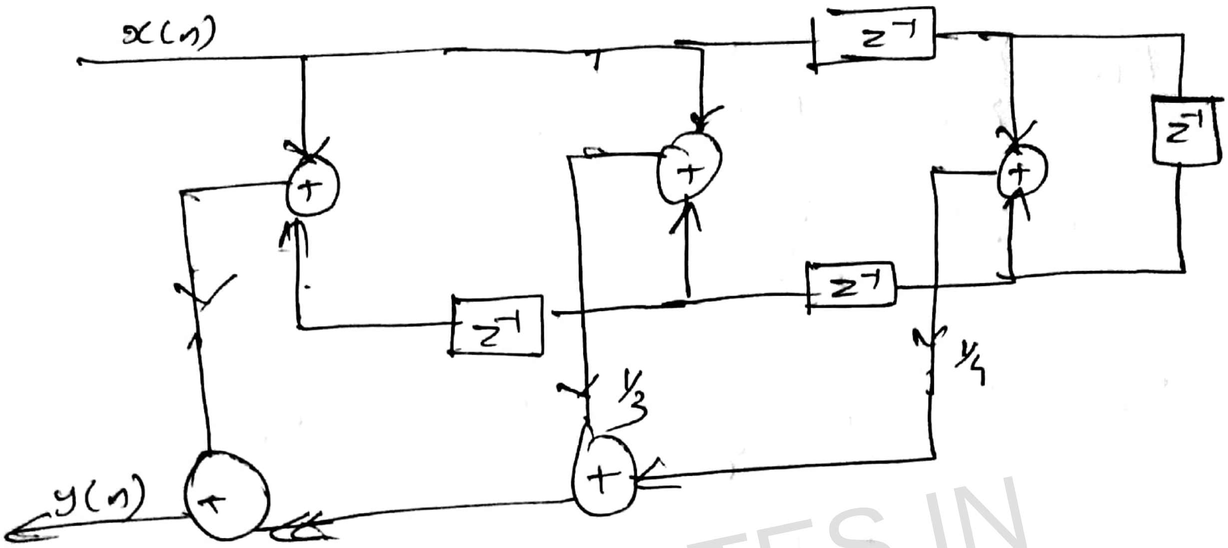
Parallel form.

$$H(z) = -3 + \frac{7}{1-0.4z^{-1}} + \frac{1}{1+0.5z^{-1}}$$



7.

b. The given H(z) has the linear phase symmetry property.



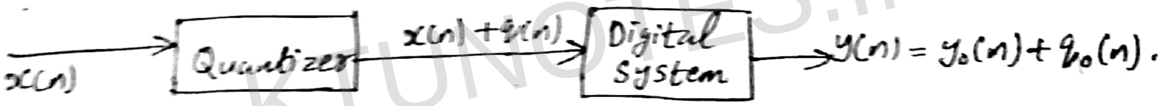
a) Finite word length effects.

When the digital filter is implemented in hardware, then, the coefficients are to be stored as binary number in register of the processor. For example, $x = 0.1110110110$ is a lengthy word. If the length of the register is 8 bit, then x is rounded (or) truncated into $x = 0.11101101$. Now, it is possible to store x value in register. But because of this truncation or rounding, filter output becomes non linear. So, limit cycle is occured.

Quantization.

Total number of bits in x is reduced by using two methods namely truncation and rounding. These are known as Quantization processes.

Quantization Noise.



Block diagram of digital system.

Here, Output of quantizer contains error signal $e(n)$. $q(n)$ is known as input noise. $y(n)$ contains output $y_0(n)$ and output noise $q_0(n)$.

The following errors arise due to quantization of numbers.

1. Input quantization error.
2. Product quantization error.
3. Coefficient quantization error.

1. The conversion of a continuous-time input signal into digital value produces an error, which is known as input quantization error. This error arises due to the represe

of the input signal by a fixed ...
conversion process.

2. Product quantization errors arise at the output of a multiplier. Multiplication of a 'b' bit data with a 'b' bit coefficient results a product having 2b bits. Since a 'b' bit register is used, the multiplier output must be rounded or truncated to 'b' bits which produces an error.

3. The filter coefficients are computed to infinite precision in theory. If they are quantized, the frequency response of the resulting filter may differ from the desired response and sometimes the filter may fail to meet the desired specification. If the poles of the desired filter are close to the unit circle, then those of the filter with quantized coefficients may lie just outside the unit circle, leading to instability.

The common methods of quantization are,

① Truncation.
② Rounding.

1) Truncation.

Truncation is a process of discarding all bits less significant than least significant bit that is retained.

eg: 0.00110011 (8 bits) to 0.0011 (4 bits).

1.01001001 (8 bits) to 1.0100 (4 bits).

When we truncate the number, the signal value is approximated by the highest quantization level that is not greater than the signal.

Rounding.

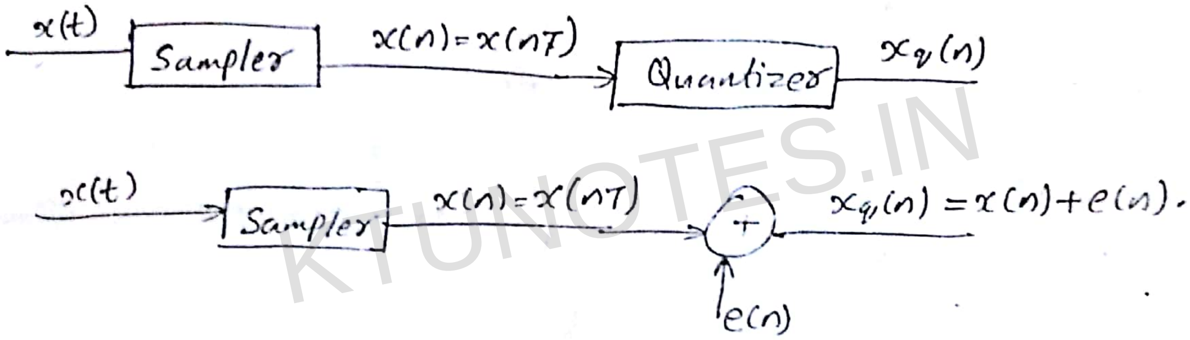
Rounding of a number of 'b' bits is accomplished by choosing the rounded result as the 'b' bit number closest to the original number unrounded.

For example 0.11010 rounded to three bits is either 0.110 or 0.111

Rounding up or down will have negligible effect on accuracy of computation.

Steady State Input Noise Power

Quantization noise model.



In digital processing of analog signals, the quantization error is commonly viewed as an additive noise signal, that is,

$$x_q(n) = x(n) + e(n).$$

$$\sigma_e^2 = \frac{2^{-2b}}{12}$$, which is also known as the steady state noise power due to input quantization.

Steady state Output Noise Power.

$$\sigma_\varepsilon^2 = \sigma_e^2 \sum_{n=0}^{\infty} h^2(n)$$

$$\sigma_\varepsilon^2 = \frac{\sigma_e^2}{2\pi j} \oint H(z) H(z^{-1}) z^{-1} dz .$$

8.b. Given $y(n) = a y(n-1) + x(n)$

$$Y(z) = a z^{-1} Y(z) + X(z)$$

$$Y(z) - a z^{-1} Y(z) = X(z)$$

$$Y(z)\left[1 - a z^{-1}\right] = X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - a z^{-1}} = \frac{z}{z-a}$$

$$H(z^{-1}) = \frac{z^{-1}}{z^{-1} - a}$$

$$\sigma_\varepsilon^2 = \sigma_e^2 \frac{1}{2\pi j} \oint_C H(z) H(z^{-1}) z^{-1} dz$$

$$= \sigma_e^2 \frac{1}{2\pi j} \oint_C \frac{z}{z-a} \cdot \frac{z^{-1}}{z^{-1} - a} z^{-1} dz$$

$$= \sigma_e^2 \frac{1}{2\pi j} \oint_C \frac{z^{-1}}{(z-a)(z^{-1}-a)} dz$$

$$\sigma_\varepsilon^2 = \sigma_e^2 \left[\text{residue of } \frac{z^{-1}}{(z-a)(z^{-1}-a)} \text{ at } z = a + \text{residue of } \frac{z^{-1}}{(z-a)(z^{-1}-a)} \text{ at } z = 1/a\right]$$

$\hookrightarrow$ equal to zero.

$$= \sigma_e^2 \left[\cancel{(z-a)} \frac{z^{-1}}{\cancel{(z-a)} (z^{-1} - a)} \right]\Bigg|_{z=a}$$

$$= \sigma_e^2 \left[\frac{1/a}{(1/a - a)} \right]$$

$$= \sigma_e^2 \left[\frac{1}{1-a^2} \right].$$

9.

a. $\sigma_\varepsilon^2 = \sigma_e^2 \frac{1}{2\pi j} \oint_C H(z) H(z^{-1}) z^{-1} dz \quad -(1)$

$H(z) = \frac{(1-a)z}{z-a}$; $H(z^{-1}) = \frac{(1-a)z^{-1}}{z^{-1}-a}$.

$\sigma_\varepsilon^2 = \sigma_e^2 \frac{1}{2\pi j} \oint_C \frac{(1-a)^2 (z^{-1}) dz}{(z-a)(z^{-1}-a)}$

$= \sigma_e^2 \Big[$ residue of $H(z) H(z^{-1}) z^{-1}$ at $z=a$

$+$ residue of $H(z) H(z^{-1}) z^{-1}$ at $z = 1/a \Big]$

$= \sigma_e^2 \left[\cancel{(z-a)} \frac{(1-a)^2 z^{-1}}{\cancel{(z-a)} (z^{-1}-a)} + 0 \right]$

$= \sigma_e^2 \left[\frac{(1-a)^2 \cdot 1/a}{(1/a - a)} \right]$

$= \sigma_e^2 \left[\frac{(1-a)^2}{a \left(\frac{1-a}{a} \right)^2} \right]$

$= \sigma_e^2 \left[\frac{(1-a)^2}{(1+a)(1-a)} \right]$

$= \sigma_e^2 \left[\frac{1-a}{1+a} \right]$

$= \frac{2^{-2b}}{12} \left[\frac{1-a}{1+a} \right]$

Q.-b.

Fixed Point Processor	Floating Point Processor
1. Fast operation	1. Slow operation.
2. Relatively economical.	2. More expensive because of costlier hardware.
3. Small dynamic range.	3. Increased dynamic range.
4. Roundoff errors occur only for addition.	4. ~~or~~ Roundoff errors can occur with both addition and multiplication
5. Overflow occur in addition.	5. Overflow does not ~~occur~~ arise.